Bioeconomic modelling of a prey predator system using differential algebraic equations

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Abstract

We propose a biological economic model based on prey-predator dynamics where the prey species are continuously harvested and predation is considered with type II functional response. The dynamic behavior of the proposed biological economic prey-predator model is discussed. Continuous type gestational delay of predators is incorporated and its effect on the dynamical behavior of the model system is analyzed. Through considering delay as a bifurcation parameter, the occurrence of Hopf bifurcation of the proposed model system with positive economic profit is shown in the neighborhood of the co-existing equilibrium point. Finally, some numerical simulations are given to verify the analytical results and the system is analyzed through graphical illustrations.

Keywords: Bioeconomics, differential algebraic system, time delay, Hopf bifurcation

1. Introduction

The application of mathematical biology has an immense impact towards the development of commonly used biological resources like fishery, wildlife and forestry. Recently Scientists and researchers give emphasis on the interaction between mathematics and biology which initiate a new research area. Interactions of mathematics and biology can be divided into three categories. The first class involves routine application of existing mathematical techniques to biological problems. Such applications influence mathematics only when the importance to biological applications requires further developments. In other cases, however, existing mathematical methods are insufficient, but it is possible to develop new mathematics within the conventional frameworks. In the final class, some fundamental issues in biology appear to require new thoughts quantitatively or analytically. Most of our biological theories evolve rapidly; therefore it is necessary to develop some useful mathematical models to describe the consequences of these biological systems. It is observed that these newly developed mathematical models are significantly influenced through the biological theories in the past and the consequent expansion of those theories in recent time. For this purpose differential algebraic equations can be considered as an important tool for the analysis of a biological model.

A general prey predator model consists of the interactions between species, therefore the model includes competition, evolution and dispersion between the species for the purpose of seeking resources to sustain their struggle for their own existence. Kar and Matsuda (2006) represented the age of maturity through time delay, which leads to systems of retarded functional differential equations. They considered a prey-predator model with Holling type of predation and harvesting of predator species and observed that when the time delay is small both the prey and predator populations reach periodic oscillations around the equilibrium in finite time then converges to their equilibrium values and in non-delay case harvesting effort has an effect of stabilizing the equilibrium. Broer *et al.* (2005) investigated a two-dimensional predator-prey model with five parameters, adapted from the Volterra-Lotka system by a non monotonic response function. They described various domains of structural stability and their bifurcations. The effect of constant rate harvesting on the dynamics of predator-prey systems has been investigated by Dai and Tang (1998), Myerscough *et al.* (1992) and Xiao and Ruan (1999), and they obtained very rich and interesting dynamical behaviours. Feng (2007) considered a differential equation system with diffusion and time delays which models the dynamics of predator-prey interactions within three biological species.

Orosz (2004) presented a formal framework for the analysis of Hopf bifurcations in delay differential equations with a single time delay. He determined closed-form linear algebraic equations and calculated the criticality of bifurcations by normal forms. Cao and Freedman (1996) obtained the criterion of persistence and global attractivity for a predator-prey model with time-delay due to gestation. Yafia et al. (2007) considered a model with one delay and a unique non trivial equilibrium. They studied the dynamics of the model in terms of the local stability and of the description of the Hopf bifurcation at non trivial equilibrium. They proved that delay (taken as a parameter of bifurcation) crosses some critical values and determined the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions. Kar (2003) studied a Gauss-type prey predator model with selective harvesting and introduce a time delay in the harvesting term. He concluded, in general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate.

Celik (2009) considered a ratio dependent predator-prey system with time delay where the dynamics is logistic with the carrying capacity proportional to prey population. Broer and Gaiko (2010) analyzed the complete global qualitative of quartic ecological model, particularly they studied the global bifurcations of singular points and limit cycles. Zhang and Zhang (2009) systematically studied a hybrid predator prey economic model, which is formulated by differential-difference-algebraic equations. They proved that this model exhibits two bifurcation phenomena at the intersampling instants. Lara and Martinet (2009) considered a discrete-time control dynamical model with uncertainties, representing a bioeconomic system, proposed through stochastic viability approach to manage natural resources in a sustainable way due to uncertainties, dynamics and conflicting objectives (ecological, social, and economical). An efficient algorithm for individual-based, stochastic simulation of biological populations in continuous time presented by Allen and Dytham (2009).

It is observed that a numerous number of research articles of the population dynamics proposed the interaction between the species and the stability analysis of the population in presence of harvesting effort but quite a few number of articles considered the bioeconomic models to investigate the dynamical behavior of the ecosystem towards the positive economic profit. Again, for the long run sustainability of the ecosystem it is necessary to compare the static as well as dynamical effects of harvesting through considering the economic perspective of the model system. Thus to formulate a biological economic system from an economic point of view and investigate the realistic static and dynamical behavior of the model system we need to use differential algebraic equations. After going through the above literature survey we can not find any biological economic model system using differential algebraic equations where prey population is harvested and the dynamical behavior of such model system is studied through considering state feedback controller.

In this paper our objective is to examine the dynamical behavior of a biological economic prey predator model where prey population is harvested using differential algebraic and bifurcation theory. The continuous gestation delay of predator population is also incorporated in the model. We have divided the paper in two parts, in the first part we consider the model system with zero economic profit and singularity induced bifurcation is obtained at the interior equilibrium of the model system. To reduce the singularity induced bifurcation, state feedback controller is designed. But in the second part we consider the model system with positive economic profit and the occurrence of Hopf bifurcation is found at the interior equilibrium point through considering delay as a bifurcation parameter. It is also proved that the time delay can cause a stable equilibrium to become unstable.

2. The model and its qualitative properties

In this section we consider a prey-predator model with Holling type of predation and continuously harvesting of prey species, the ecological set up of which is as follows. It is assumed that the predator is not harvested and hence harvesting does not affect the growth of the predator population directly. However, it is considered that the predators have competition among themselves for their survival. Again there exists conflict between predators and harvesters for common resource i.e. prey species. The growth of prey is assumed to be logistic. Let us assume x and y are respectively the size of the prey and predator population at time t. Thus, the consequent model becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - h(t),$$

$$\frac{dy}{dt} = \frac{\beta yx}{a+x} - dy - \gamma y^2, \qquad\qquad (1)$$

where r is the intrinsic growth rate of the prey, K is the environmental carrying capacity of prey, α is the maximal relative increase of predation, a is Michaelis-Menten constant, $h(t)$ is the harvesting at time t, d is the death rate of predator, the predator consumes prey at the rate β (we assume $0 < \beta < 1$, since the whole biomass of the prey is not transformed to the biomass of the predator).

Density dependent mortality rate γy^2 describes either a self limitation of consumers or the influence of predation. γ is the intra-specific coefficient of the predator population. Self limitation can occur if there is some other factor (other than food) which becomes limiting at high population densities.

The functional form of harvest is generally considered using the phrase catch-per-unit-effort (CPUE) hypothesis (Clark, 1990) to describe an assumption that catch per unit effort is proportional to the stock level. Thus we consider

$$h(t) = qEx, \tag{2}$$

where E is the harvesting effort used to harvest prey population and q is the catchability co-efficient of prey population.

The Antarctic krill-whale community is a good example of the present model. Krill is a main source of food of whales and the Antarctic krill population is being increasingly harvested. On the other hand, the moratorium imposed by IWC on killing of whales continues. Large catches from the lower trophic level (krill) can have serious implications for production at both the lower trophic level (krill) and the higher trophic level (whale). It is, therefore, necessary to regulate harvesting at the lower trophic level.
Let us extend our model by considering the following algebraic equation

$$(pqx - c)E - s = 0, \tag{3}$$

where c is the constant fishing cost per unit effort, p is the constant price per unit biomass of landed fish and s is the total economic rent obtained from the fishery.

Thus, using (2) & (3) system (1) becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + x} - qEx,$$

$$\frac{dy}{dt} = \frac{\beta yx}{a + x} - dy - \gamma y^2, \tag{4}$$

$$(pqx - c)E - s = 0,$$

Let us now consider this harvested prey predator system with continuous time delay due to gestation. Here the predator population consumes the prey population at a constant rate β, but the reproduction of predators after predating the prey population is not instantaneous thus it will be incorporated by some time lag required for gestation of predators. Suppose the time interval between the moments when an individual prey is killed and the corresponding biomass is added to the predator population is considered as the time delay τ. Let us take the entire past history of prey biomass, which is to be measured by $\frac{1}{\tau}\exp\left(-\frac{1}{\tau}(t - \tau_0)\right)$, where $\tau_0 < t$ is considered as a particular time in the past and t represents the present time. Thus the prey biomass in predator's equation is replaced by the following form

$$z(t) = \int_{-\infty}^{t} x(t)\frac{1}{\tau}\exp\left(-\frac{1}{\tau}(t - \tau_0)\right) d\tau_0. \tag{5}$$

Under this assumption the final system becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + x} - qEx,$$

$$\frac{dy}{dt} = \frac{\beta yz}{a + z} - dy - \gamma y^2,$$

$$\frac{dz}{dt} = \frac{1}{\tau}(x - z), \tag{6}$$

$$(pqx - c)E - s = 0.$$

The differential algebraic system (6) can be expressed in the following way,

$$f(X,E,s) = \begin{bmatrix} f_1(X,E,s) \\ f_2(X,E,s) \\ f_3(X,E,s) \end{bmatrix} = \begin{bmatrix} rx\left(1 - \dfrac{x}{K}\right) - \dfrac{\alpha xy}{a+x} - qEx \\ \dfrac{\beta yx}{a+x} - dy - \gamma y^2 \\ \dfrac{1}{\tau}(x-z) \end{bmatrix}$$

$$g(X,E,s) = (pqx\text{-}c)E\text{-}s = 0,$$

where $X = (x, y, z)^t.$

Let us now consider two cases separately with zero economic profit and with positive economic profit.

3. The model with zero economic profit

For $s = 0,$ the system (6) becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - qEx,$$

$$\frac{dy}{dt} = \frac{\beta yz}{a+z} - dy - \gamma y^2,$$

(7)

$$\frac{dz}{dt} = \frac{1}{\tau}(x-z),$$

$$(pqx\text{-}c)E = 0.$$

Equilibrium points: existence and stability
The following lemma represents all possible non negative equilibrium points of system (7).

Lemma 1
System (7) has two equilibrium points $P_0\,(0,0,0,0)$ and $P_1(K,0,K,0)$ for any positive set of parameters. The third boundary equilibrium point $P_2\,(\tilde{x},0,\tilde{z},\tilde{E})$ exists if and only if $c < Kpq$ where $\tilde{x} = \dfrac{c}{pq}, \tilde{z} = \dfrac{c}{pq}, \quad \tilde{E} = \dfrac{-cr + Kpqr}{Kpq^2}.$ The interior equilibrium point $P_3\,(x^*, y^*, z^*, E^*)$ of the system (7) exists if $c\beta > cd + adpq$ and $r + \dfrac{pq\alpha(cd + adpq - c\beta)}{(c+apq)^2\gamma} > \dfrac{cr}{Kpq}.$ When these conditions are satisfied x^*, y^*, z^* and

E^* are given by $x^* = \dfrac{c}{pq}, y^* = \dfrac{-cd - adpq + c\beta}{(c+apq)\gamma}, z^* = \dfrac{c}{pq}, E^* = \dfrac{1}{q}\left(r - \dfrac{cr}{Kpq} - \dfrac{\alpha(-cd - adpq + c\beta)}{\left(a + {c}/{pq}\right)(c+apq)\gamma}\right).$

From system (7) we have the following matrix

$$M = D_X f - D_E f (D_E g)^{-1} D_X g$$

$$= \begin{bmatrix} r - \dfrac{2xr}{K} - \dfrac{a\alpha y}{(a+x)^2} - qE + \dfrac{pq^2 Ex}{pqx-c} & \dfrac{-\alpha x}{a+x} & 0 \\ 0 & \dfrac{\beta z}{a+z} - d - 2y\gamma & \dfrac{a\beta y}{(a+z)^2} \\ \dfrac{1}{\tau} & 0 & -\dfrac{1}{\tau} \end{bmatrix}.$$

The characteristic polynomial of the matrix M is given by

$$\mu^3 + a_1(X,E)\mu^2 + a_2(X,E)\mu + a_3(X,E) = 0,$$

where

$$a_1 = d + \tfrac{1}{\tau} + r(-1 + \tfrac{2x}{K}) + \tfrac{cEq}{c-pqx} + \frac{ay\alpha}{(a+x)^2} - \tfrac{z\beta}{a+z} + 2y\gamma,$$

$$a_2 = -\frac{r}{\tau} + \frac{2rx}{K\tau} + \frac{ay\alpha}{\tau(a+x)^2} + d\left[\frac{1}{\tau} + r\left(-1 + \frac{2x}{K}\right) + \frac{cEq}{c-pqx} + \frac{ay\alpha}{(a+x)^2}\right] + \frac{rz\beta}{a+z} - \frac{z\beta}{\tau(a+z)} - \frac{2rxz\beta}{K(a+z)} - \frac{ayz\alpha\beta}{(a+x)^2(a+z)}$$

$$- 2ry\gamma + \frac{2y\gamma}{\tau} + \frac{4rxy\gamma}{K} + \frac{2ay^2\alpha\gamma}{(a+x)^2} + \frac{cEq(a+z-\tau\beta z + 2a\tau y\gamma + 2\tau yz\gamma)}{\tau(c-pqx)(a+z)},$$

$$a_3 = \frac{1}{\tau}\left[d\left(r(-1+\tfrac{2x}{K}) + \frac{cEq}{c-pqx} + \frac{ay\alpha}{(a+x)^2}\right) + \frac{rz\beta}{a+z} - \frac{2rxz\beta}{K(a+z)} + \frac{axy\alpha\beta}{(a+x)(a+z)^2} - \frac{ayz\alpha\beta}{(a+x)^2(a+z)} - 2ry\gamma + \frac{4rxy\gamma}{K}\right.$$

$$\left. + \frac{2ay^2\alpha\gamma}{(a+x)^2} + \frac{cEq(-\beta z + 2y(a+z)\gamma)}{(c-pqx)(a+z)}\right].$$

The stability of the boundary equilibrium points $P_0(0,0,0,0)$ and $P_1(K,0,K,0)$ of system (7) is given in the following lemma.

Lemma 2

The boundary equilibrium point $P_0(0,0,0,0)$ is unstable and $P_1(K,0,K,0)$ is locally asymptotically stable if $\beta < d\left(1 + \dfrac{a}{K}\right)$.

The eigen values of the characteristic polynomial at the boundary equilibrium point $P_0(0,0,0,0)$ are $-d, r, \dfrac{1}{\tau}$. Thus the boundary equilibrium point $P_0(0,0,0,0)$ is unstable. Again the eigen values of the characteristic polynomial at the boundary equilibrium point $P_1(K,0,K,0)$ are $-r, -\dfrac{1}{\tau}, \dfrac{-ad - dK + K\beta}{a + K}$. It is clearly observed that the boundary equilibrium point $P_1(K,0,K,0))$ is stable if $\beta < d\left(1 + \dfrac{a}{K}\right)$.

Let us now study the dynamic behavior of the differential algebraic model system (7). The local stability of the boundary equilibrium point, $P_2(\tilde{x},0,\tilde{z},\tilde{E})$ and the interior equilibrium point, $P_3(x^*, y^*, z^*, E^*)$ can be investigated using the singularity induced bifurcation phenomena. Here, we are interested to discuss the local stability of the model system (7) at the interior equilibrium point, $P_3(x^*, y^*, z^*, E^*)$ through bifurcation phenomena. For this purpose total economic rent is assumed to be the bifurcation parameter i.e., $\mu = s$.

Consequently we have the following theorem

Theorem 1

The differential algebraic system (7) has a singularity induced bifurcation at the interior equilibrium point $P_3(x^*, y^*, z^*, E^*)$. When the bifurcation parameter s increases through zero the stability of the interior equilibrium point $P_3(x^*, y^*, z^*, E^*)$ changes from stable to unstable.

Proof.   It is evident that $D_E g = pqx - c$ has a simple zero eigen value. Thus, we can define $\Delta(X, E, s) = D_E g = pqx - c$.

(i) It follows from Lemma1 that

$$trace(D_E f\ adj(D_E g)D_X g)_{P_3} = -pq^2 x^* E^* \neq 0.$$

(ii) It can be proved using Lemma1 that

$$\begin{vmatrix} D_X f & D_E f \\ D_X g & D_E g \end{vmatrix}_{P_3} = \begin{vmatrix} r - \frac{2xr}{K} - \frac{a\alpha y}{(a+x)^2} - qE & \frac{-\alpha x}{a+x} & 0 & -qx \\ 0 & \frac{\beta z}{a+z} - d - 2y\gamma & \frac{a\beta y}{(a+z)^2} & 0 \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} & 0 \\ pqE & 0 & 0 & pqx - c \end{vmatrix}_{P_3}$$

$$= \frac{pq^2 x^* y^* E^* \gamma}{\tau} \neq 0.$$

(iii) It can also be shown using Lemma1 that

$$\begin{vmatrix} D_X f & D_E f & D_\lambda f \\ D_X g & D_E g & D_\lambda g \\ D_X \Delta & D_E \Delta & D_\lambda \Delta \end{vmatrix}_{P_3} = \begin{vmatrix} r - \frac{2xr}{K} - \frac{a\alpha y}{(a+x)^2} - qE & \frac{-\alpha x}{a+x} & 0 & -qx & 0 \\ 0 & \frac{\beta z}{a+z} - d - 2y\gamma & \frac{a\beta y}{(a+z)^2} & 0 & 0 \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} & 0 & 0 \\ pqE & 0 & 0 & pqx - c & -1 \\ pq & 0 & 0 & 0 & 0 \end{vmatrix}_{P_3}$$

$$= \frac{pq^2 x^* y^* \gamma}{\tau} \neq 0.$$

It is observed from (i)-(iii) that all the conditions for singularity induced bifurcation (Venkatasubramanian *et al.*, 1995) are satisfied. Hence the differential algebraic system (7) has a singularity induced bifurcation at the interior equilibrium point $P_3(x^*, y^*, z^*, E^*)$ and the bifurcation value is s=0.

Again, it is noted that

$$M_1 = -trace(D_E f \, adj(D_E g) D_X g)_{P_3} = pq^2 x^* E^*,$$

$$M_2 = \left[D_\lambda \Delta - (D_X \Delta \quad D_E \Delta) \begin{pmatrix} D_X f & D_E f \\ D_X g & D_E g \end{pmatrix}^{-1} \begin{pmatrix} D_\lambda f \\ D_\lambda g \end{pmatrix} \right]_{P_3} = \frac{1}{E^*}.$$

Consequently from Lemma1 we have,

$$\frac{M_1}{M_2} = pq^2 x^* E^{*2} > 0.$$

Hence, it can be concluded from (Venkatasubramanian *et al.*, 1995) that when s increases through zero, one eigenvalue of the model system (7) moves from C^- to C^+ along the real axis by diverging through ∞. Consequently the stability of the model system (7) is influenced through this behavior i.e., the stability of the system at the interior equilibrium point $P_3(x^*, y^*, z^*, E^*)$ changes from stable to unstable.

In consequence to the above theorem it is clear that the differential algebraic model system (6) becomes unstable when the economic interest of the harvesting is considered to be positive. If we consider economic perspective of the fishery it is obvious that fishery agencies are interested towards the positive economic rent earned from the fishery. It is also noted that an impulsive phenomenon can occur through singularity induced bifurcation in prey predator ecosystem which may lead to the collapse of the sustainable ecosystem of the prey predator fishery.

Therefore, it is necessary to reduce the impulsive phenomenon from the prey predator ecosystem to resume the sustainability of the ecosystem and stabilize the model system when positive economic interest is considered for fishery managers.

Thus, to stabilize the model system (6) in case of positive economic interest a state feedback controller, (Dai, 1989) can be designed of the form $w(t) = u(E(t) - E^*)$,

where u stands for net feedback gain.

Let us introduce the state feedback controller to the model system (6) and rewrite model system as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - qEx,$$

$$\frac{dy}{dt} = \frac{\beta yz}{a+z} - dy - \gamma y^2,$$

(8)

$$\frac{dz}{dt} = \frac{1}{\tau}(x-z),$$

$$(pqx - c)E - s + u(E(t) - E^*) = 0.$$

Consequently, we have the following theorem:

Theorem 2

The differential algebraic model system (8) is stable at the interior equilibrium point, $P_3(x^*, y^*, z^*, E^*)$, of the model system (7), if

$$u > \max\left[\frac{pq^2 x^* E^*}{\dfrac{rx^*}{K} - \dfrac{x^* y^* \alpha}{(a+x^*)^2} + y^*\gamma + \dfrac{1}{\tau}}, \quad \frac{pq^2 x^* y^* E^* \gamma + \dfrac{pq^2 x^* E^*}{\tau}}{\dfrac{rx^* y^* \gamma}{K} - \dfrac{x^* y^{*2} \alpha\gamma}{(a+x^*)^2} + \dfrac{rx^*}{K\tau} - \dfrac{x^* y^* \alpha}{(a+x^*)^2 \tau} + \dfrac{y^* \gamma}{\tau}}, \right.$$

$$\left. \frac{\dfrac{pq^2 x^* y^* E^* \gamma}{\tau}}{\dfrac{ax^* y^* \alpha\beta}{(a+x^*)(a+z^*)^2 \tau} + \dfrac{rx^* y^* \gamma}{K\tau} - \dfrac{x^* y^{*2} \alpha\gamma}{(a+x^*)^2 \tau}} \right].$$

Proof. For the differential algebraic model system (8), we can obtain the following Jacobian at the interior equilibrium point $P_3(x^*, y^*, z^*, E^*)$, of the model system (7),

$$J_{P_3} = \begin{bmatrix} \dfrac{x^* y^* \alpha}{(a+x^*)^2} - \dfrac{rx^*}{K} + \dfrac{pq^2 x^* E^*}{u} & \dfrac{-\alpha x^*}{a+x^*} & 0 \\[2mm] 0 & -\gamma y^* & \dfrac{a\beta y^*}{(a+z^*)^2} \\[2mm] \dfrac{1}{\tau} & 0 & -\dfrac{1}{\tau} \end{bmatrix}.$$

Therefore, the characteristic polynomial of the matrix J is given

$$\mu^3 + w_1(X,E)\mu^2 + w_2(X,E)\mu + w_3(X,E) = 0,$$

where

$$w_1 = \frac{rx^*}{K} - \frac{pq^2 x^* E^*}{u} - \frac{x^* y^* \alpha}{(a+x^*)^2} + y^* \gamma + \frac{1}{\tau},$$

$$w_2 = \frac{rx^* y^* \gamma}{K} - \frac{pq^2 x^* y^* E^* \gamma}{u} - \frac{x^* y^{*2} \alpha\gamma}{(a+x^*)^2} + \frac{rx^*}{K\tau} - \frac{pq^2 x^* E^*}{u\tau} - \frac{x^* y^* \alpha}{(a+x^*)^2 \tau} + \frac{y^* \gamma}{\tau},$$

$$w_3 = \frac{ax^* y^* \alpha\beta}{(a+x^*)(a+z^*)^2 \tau} + \frac{rx^* y^* \gamma}{K\tau} - \frac{pq^2 x^* y^* E^* \gamma}{u\tau} - \frac{x^* y^{*2} \alpha\gamma}{(a+x^*)^2 \tau}.$$

According to the Routh Hurwitz criterion it can be concluded that the model system (8) is stable at the interior equilibrium point, $P_3(x^*, y^*, z^*, E^*)$, of the model system (7) if the net feedback gain u satisfies the following condition:

$$u > \max \left[\frac{pq^2 x^* E^*}{\dfrac{rx^*}{K} - \dfrac{x^* y^* \alpha}{(a+x^*)^2} + y^* \gamma + \dfrac{1}{\tau}}, \ \frac{pq^2 x^* y^* E^* \gamma + \dfrac{pq^2 x^* E^*}{\tau}}{\dfrac{rx^* y^* \gamma}{K} - \dfrac{x^* y^{*2} \alpha \gamma}{(a+x^*)^2} + \dfrac{rx^*}{K\tau} - \dfrac{x^* y^* \alpha}{(a+x^*)^2 \tau} + \dfrac{y^* \gamma}{\tau}}, \right.$$

$$\left. \frac{\dfrac{pq^2 x^* y^* E^* \gamma}{\tau}}{\dfrac{ax^* y^* \alpha \beta}{(a+x^*)(a+z^*)^2 \tau} + \dfrac{rx^* y^* \gamma}{K\tau} - \dfrac{x^* y^{*2} \alpha \gamma}{(a+x^*)^2 \tau}} \right].$$

Hence, it is possible to eliminate singularity induced bifurcation which is responsible for impulsive phenomenon in a sustainable ecosystem, using a suitably designed net feedback gain. Again, the economic interest of fishery managers can also be achieved using the state feedback controller function i.e., the stability can be resumed for the model system (8) when positive economic interest is considered.

4. The model with positive economic profit

In this section we consider the model system with positive economic profit i.e., $s \neq 0$. Here we investigate the system behavior for two separate cases with and without time delay.

4.1 The model without time delay
The model system (6) without time delay can be written as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a+x} - qEx,$$

$$\frac{dy}{dt} = \frac{\beta xy}{a+x} - dy - \gamma y^2, \tag{9}$$

$$(pqx - c)E - s = 0.$$

The interior equilibrium point of the system (9) is $\overline{P}(\overline{x}, \overline{y}, \overline{E})$

where $\overline{y} = \dfrac{\beta \overline{x} - d(a+\overline{x})}{\gamma(a+\overline{x})}, \ \overline{E} = \dfrac{s}{pq\overline{x} - c}$ and $\overline{x}$ satisfies the following equation

$$C_0 \overline{x}^4 + C_1 \overline{x}^3 + C_2 \overline{x}^2 + C_3 \overline{x} + C_4 = 0, \tag{10}$$

where $C_0 = pqr\gamma,$

$C_1 = -cr\gamma + 2apqr\gamma - Kpqr\gamma,$

$C_2 = -dKpq\alpha + Kpq\alpha\beta - 2acr\gamma + cKr\gamma + a^2 pqr\gamma - 2aKpqr\gamma + Kqs\gamma,$

$C_3 = cdK\alpha - adKpq\alpha - cK\alpha\beta - a^2 cr\gamma + 2acKr\gamma - a^2 Kpqr\gamma + 2aKqs\gamma,$

$C_4 = acdK\alpha + a^2 cKr\gamma + a^2 Kqs\gamma.$

From system (9) we have the following matrix

$$N = \begin{bmatrix} \dfrac{\overline{x}\,\overline{y}\alpha}{(a+\overline{x})^2} - \dfrac{r\overline{x}}{K} + \dfrac{pq^2 \overline{x}\overline{E}}{pq\overline{x} - c} & \dfrac{-\alpha \overline{x}}{a+\overline{x}} \\[3mm] \dfrac{a\beta\,\overline{y}}{(a+\overline{x})^2} & -\gamma\,\overline{y} \end{bmatrix}.$$

Thus, the characteristic polynomial of the matrix N at $\overline{P}(\overline{x}, \overline{y}, \overline{E})$ is given by

$$\mu^2 + b_1(X, E)\mu + b_2(X, E) = 0,$$

where

$$b_1 = \frac{r\overline{x}}{K} - \frac{\overline{x}\,\overline{y}\alpha}{(a+\overline{x})^2} - \frac{pq^2\overline{x}\overline{E}}{pq\overline{x} - c} + \gamma\,\overline{y},$$

$$b_2 = \frac{a\overline{x}\,\overline{y}\alpha\beta}{(a+\overline{x})^3} + \frac{r\overline{x}\,\overline{y}\gamma}{K} - \frac{\overline{x}\,\overline{y}^2\alpha\gamma}{(a+\overline{x})^2} - \frac{pq^2\overline{x}\overline{E}\,\overline{y}\gamma}{pq\overline{x} - c}.$$

We find that $b_1 > 0$ and $b_2 > 0$ if $\dfrac{r}{K} > \dfrac{pq^2\overline{E}}{pq\overline{x} - c} + \dfrac{\overline{y}\alpha}{(a+\overline{x})^2}.$

Hence, the interior equilibrium point $\overline{P}(\overline{x}, \overline{y}, \overline{E})$ of system (9) is asymptotically stable if

$$\frac{r}{K} > \frac{pq^2\overline{E}}{pq\overline{x} - c} + \frac{\overline{y}\alpha}{(a+\overline{x})^2}.$$

In particular, if we consider intra-specific coefficients of the predator population is zero i.e., $\gamma = 0$ then the interior equilibrium of the model system (9) becomes $P(x^*, y^*, E^*)$,

where

$$x^* = \frac{ad}{\beta - d},$$

$$y^* = \frac{1}{\alpha}\left(ar - \frac{a^2 d^2 r}{K(d-\beta)^2} - \frac{adr}{d-\beta} + \frac{a^2 dr}{K(d-\beta)} - \frac{aq(-ds+s\beta)}{cd+adpq-c\beta} + \frac{adq(-ds+s\beta)}{(d-\beta)(cd+adpq-c\beta)} \right) \text{ and}$$

$$E^* = \frac{s\beta - ds}{cd + adpq - c\beta}.$$

It is noted that for the existence of the interior equilibrium point, $P(x^*, y^*, E^*)$, it is necessary $\beta > d$, $cd + adpq > c\beta$ and

$$rK > \frac{adr}{\beta - d} + \frac{Kps(d-\beta)^2}{(\beta-d)(cd+adpq-c\beta)}.$$

In this particular case the characteristic polynomial of the matrix N at $P(x^*, y^*, E^*)$, is reduced to,

$$\mu^2 + p_1(X, E)\mu + p_2(X, E) = 0, \qquad\qquad (11)$$

where

$$p_1 = \frac{rx^*}{K} - \frac{x^* y^* \alpha}{(a+x^*)^2} - \frac{pq^2 x^* E^*}{pqx^* - c},$$

$$p_2 = \frac{ax^* y^* \alpha\beta}{(a+x^*)^3} > 0.$$

We find that $\quad p_1 > 0$ if $\dfrac{r}{K} > \dfrac{pq^2 E^*}{pqx^* - c} + \dfrac{y^* \alpha}{(a+x^*)^2}.$

Hence, the interior equilibrium point $P(x^*, y^*, E^*)$, of system (9) is asymptotically stable if $\dfrac{r}{K} > \dfrac{pq^2 E^*}{pqx^* - c} + \dfrac{y^* \alpha}{(a+x^*)^2}.$

Again, it is observed that for $s = s^* = -\dfrac{r(adpq + c(d - \beta))^2 (K(d - \beta) + a(d + \beta))}{Kq(c + apq)(d - \beta)^3}$, $p_1 = 0$ and hence the roots of the characteristic equation (11) become purely imaginary and they are conjugate to each other. Also we have

$$\frac{d}{ds}\left[traceN_{P(x^*,y^*,E^*)}\right]_{s=s^*} = \frac{dq(c + apq)(d - \beta)^2}{(adpq + c(d - \beta))^2 \beta} \neq 0.$$

Hence by the Hopf bifurcation theorem (Hassard et al., 1981), the system (9) enters into a Hopf type small amplitude periodic solution at $s = s^*$ in absence of γ near the positive interior equilibrium point $P(x^*, y^*, E^*)$.

4.2 The model with time delay

In this section we consider the model system (6). It is evident that the coordinates of the interior equilibrium point, $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ of model system (6) is as follows: $\hat{y} = \bar{y}, \hat{E} = \bar{E}$ and $\hat{x} = \hat{z}$ where $\hat{x}$ satisfying the equation (10) thus $\hat{x}$ can be evaluated from equation (10).

From the model system (6) we have the following matrix

$$R = \begin{bmatrix} \dfrac{\alpha \hat{x} \hat{y}}{(a + \hat{x})^2} - \dfrac{r\hat{x}}{K} + \dfrac{pq^2 \hat{E}\hat{x}}{pq\hat{x} - c} & \dfrac{-\alpha \hat{x}}{a + \hat{x}} & 0 \\[3mm] 0 & -\hat{y}\gamma & \dfrac{a\beta \hat{y}}{(a + \hat{z})^2} \\[3mm] \dfrac{1}{\tau} & 0 & -\dfrac{1}{\tau} \end{bmatrix}.$$

Thus, the characteristic polynomial of the matrix R is given by

$$\mu^3 + d_1(X, E)\mu^2 + d_2(X, E)\mu + d_3(X, E) = 0,$$

where

$$d_1 = \frac{r\hat{x}}{K} - \frac{\alpha \hat{x} \hat{y}}{(a + \hat{x})^2} - \frac{pq^2 \hat{E}\hat{x}}{pq\hat{x} - c} + \hat{y}\gamma + \frac{1}{\tau},$$

$$d_2 = \frac{r\hat{x}}{K\tau} - \frac{pq^2 \hat{E}\hat{x}}{(pq\hat{x} - c)\tau} + \frac{r\hat{x}\hat{y}\gamma}{K} - \frac{\alpha \hat{x}\hat{y}}{(a + \hat{x})^2 \tau} - \frac{pq^2 \hat{E}\hat{x}\hat{y}\gamma}{pq\hat{x} - c} + \frac{\hat{y}\gamma}{\tau} - \frac{\alpha \hat{x}\hat{y}^2 \gamma}{(a + \hat{x})^2},$$

$$d_3 = \frac{a\hat{x}\hat{y}\alpha\beta}{\tau(a + \hat{x})(a + \hat{z})^2} + \frac{r\hat{x}\hat{y}\gamma}{K\tau} - \frac{\alpha \hat{x}\hat{y}^2 \gamma}{\tau(a + \hat{x})^2} - \frac{pq^2 \hat{E}\hat{x}\hat{y}\gamma}{(pq\hat{x} - c)\tau}.$$

It is noted that $d_1 > 0$ and $d_3 > 0$ if $\dfrac{r}{K} > \dfrac{pq^2 \hat{E}}{pq\hat{x} - c} + \dfrac{\hat{y}\alpha}{(a + \hat{x})^2}$.

Let us assume $A(\tau) = d_1 d_2 - d_3$.

Then, $A(\tau) = \dfrac{1}{\tau^2}(t_1 \tau^2 + t_2 \tau + t_3),$

where

$$t_1 = \frac{r^2 \hat{x}^2 \hat{y}\gamma}{K^2} + \frac{p^2 q^4 \hat{E}^2 \hat{x}^2 \hat{y}\gamma}{(pq\hat{x}-c)^2} - \frac{2pq^2 r\hat{E}\hat{x}^2 \hat{y}\gamma}{(pq\hat{x}-c)K} - \frac{2r\alpha\gamma \hat{x}^2 \hat{y}^2}{(a+\hat{x})^2 K} + \frac{2pq^2 \hat{E}\hat{x}^2 \hat{y}^2 \alpha\gamma}{(a+\hat{x})^2 (pq\hat{x}-c)} + \frac{\alpha^2 \hat{x}^2 \hat{y}^3 \gamma}{(a+\hat{x})^4}$$

$$+ \frac{r\hat{x}\hat{y}^2 \gamma^2}{K} - \frac{pq^2 \hat{E}\hat{x}\hat{y}^2 \gamma^2}{pq\hat{x}-c} - \frac{\alpha \hat{x}\hat{y}^3 \gamma^2}{(a+\hat{x})^2},$$

$$t_2 = \frac{r^2 \hat{x}^2}{K^2} + \frac{p^2 q^4 \hat{E}^2 \hat{x}^2}{(pq\hat{x}-c)^2} - \frac{2pq^2 r\hat{E}\hat{x}^2}{(pq\hat{x}-c)K} - \frac{2r\alpha \hat{x}^2 \hat{y}}{(a+\hat{x})^2 K} + \frac{2pq^2 \hat{E}\hat{x}^2 \hat{y}\alpha}{(a+\hat{x})^2 (pq\hat{x}-c)} + \frac{\alpha^2 \hat{x}^2 \hat{y}^2}{(a+\hat{x})^4} - \frac{a\alpha \beta \hat{x}\hat{y}}{(a+\hat{x})(a+\hat{z})^2}$$

$$+ \frac{2r\hat{x}\hat{y}\gamma}{K} - \frac{2pq^2 \hat{E}\hat{x}\hat{y}\gamma}{pq\hat{x}-c} - \frac{2\alpha \hat{x}\hat{y}^2 \gamma}{(a+\hat{x})^2} + \hat{y}^2 \gamma^2,$$

$$t_3 = \frac{r\hat{x}}{K} - \frac{\alpha \hat{x}\hat{y}}{(a+\hat{x})^2} - \frac{pq^2 \hat{E}\hat{x}}{pq\hat{x}-c} + \hat{y}\gamma.$$

Now we have the following theorem which ensures the local stability of the interior equilibrium point, $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ of the model system (6).

Theorem 3

If $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ exists with $\frac{r}{K} > \frac{pq^2 \hat{E}}{pq\hat{x}-c} + \frac{\hat{y}\alpha}{(a+\hat{x})^2}$ and $t_1\tau^2 + t_2\tau + t_3 > 0$, then $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ is locally asymptotically stable.

Proof. The condition $\frac{r}{K} > \frac{pq^2 \hat{E}}{pq\hat{x}-c} + \frac{\hat{y}\alpha}{(a+\hat{x})^2}$ implies that $d_1 > 0$ and $d_3 > 0$. Finally, $t_1\tau^2 + t_2\tau + t_3 > 0$ implies that $A(\tau) = d_1 d_2 - d_3 > 0$. Hence by Routh Hurwitz criterion, the theorem follows.

Bifurcation analysis

Prey-predator models with constant parameters are often found to approach a steady state in which the species coexist in equilibrium. But if parameters used in the model are changed, other types of dynamical behavior may occur and the critical parameter values at which such transitions happen are called bifurcation points.
Now we analyze the bifurcation of the model system (6) assuming τ as the bifurcation parameter.

Theorem 4

If $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ exists with $\frac{r}{K} > \frac{pq^2 \hat{E}}{pq\hat{x}-c} + \frac{\hat{y}\alpha}{(a+\hat{x})^2}$ and $\tau < \frac{2\gamma(a+\hat{x})^3}{a\hat{x}\alpha\beta}$, then a simple Hopf bifurcation occurs at the positive unique value $\tau = \tau^*$.

Proof. The characteristic equation of the model system (6) at $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ is given by

$$\mu^3 + d_1(\tau)\mu^2 + d_2(\tau)\mu + d_3(\tau) = 0, \tag{12}$$

The equation (12) has two purely imaginary roots if and only if $d_1 d_2 = d_3$ for a unique value of τ (say τ^*) at which we have a Hopf bifurcation. Thus in the neighborhood of τ^* the characteristic equation (12) can't have real roots. For $\tau = \tau^*$ we have $(\mu^2 + d_2)(\mu + d_1) = 0$. This equation has two purely imaginary roots and a real root as,
$\mu_1 = i\sqrt{d_2}, \mu_2 = -i\sqrt{d_2}$ and $\mu_3 = -d_1$.
The roots are of the following form

$$\mu_1(\tau) = p(\tau) + iq(\tau), \mu_2(\tau) = p(\tau) - iq(\tau) \text{ and } \mu_3(\tau) = -d_1(\tau).$$

To apply Hopf bifurcation theorem as stated in Liu's criterion (Liu,1994) we need to verify the transversality condition,

$$\left[\frac{dp(\tau)}{d\tau}\right]_{\tau=\tau^*} \neq 0.$$

Substituting $\mu_1(\tau) = p(\tau) + iq(\tau)$, in the equation (12) and differentiating the resulting equation w.r.t. τ, and setting $p(\tau) = 0$ and $q(\tau) = \sqrt{d_2}$, we get the transversality condition at $\tau = \tau^*$ as

$$\left[\frac{dp(\tau)}{d\tau}\right]_{\tau=\tau^*} = \left[-\frac{d_2(d_1 d_2' - d_3' + d_1' d_2)}{2(d_2^2 + d_1^2 d_2)}\right]_{\tau=\tau^*}.$$

Thus from the expressions of d_1, d_2 and d_3 we find,

$$\left[\frac{dp(\tau)}{d\tau}\right]_{\tau=\tau^*} = \frac{v^2 \hat{x}^2 \tau^* + 2v(\hat{x} + \hat{x}\hat{y}\gamma\tau^*) + \hat{y}\left(2\gamma - \dfrac{a\hat{x}\alpha\beta\tau^*}{(a+\hat{x})(a+\hat{z})^2} + \hat{y}\gamma^2\tau^*\right)}{2\left(v\hat{x}\hat{y}\gamma + \left(v\hat{x} + \hat{y}\gamma + \dfrac{1}{\tau^*}\right)^2 + \dfrac{v\hat{x}}{\tau^*} + \dfrac{\hat{y}\gamma}{\tau^*}\right)\tau^{*3}} > 0 \text{ if } \tau < \frac{2\gamma(a+\hat{x})^3}{a\hat{x}\alpha\beta},$$

where $\quad v = \dfrac{r}{K} - \dfrac{pq^2 \hat{E}}{pq\hat{x}-c} - \dfrac{\hat{y}\alpha}{(a+\hat{x})^2}.$

Thus, it can be concluded that the interior equilibrium point, $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ is locally asymptotically stable for $\tau < \tau^*$. Furthermore, according to the Liu's criterion a simple Hopf bifurcation occurs at $\tau = \tau^*$ and for $\tau > \tau^*$, the interior equilibrium point, $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ approaches to a periodic solution.
Hence the theorem follows.

5. Numerical simulations and discussion

In this section we assign numerical values to the parameters of the model system (6) and compute some simulations using those values. For the purpose of simulation experiments we mainly use the software MATLAB 7.0 and MATHEMATICA 5.2. This section can be classified into two categories. First category consists of the results where the total economic profit is considered to be zero. In the second category, numerical simulations are represented with positive economic profit.

5.1 Simulation when total economic profit is zero
In order to ensure the analytical result of theorem1 numerically let us assign the following numerical values to the parameters of the model system (6); $\alpha = 0.95$, $r = 2$, $\beta = 0.75$, $K = 100$, $q = 0.5$, $a = 30$, $d = 0.002$, $\gamma = 0.05$, $p = 15$, $c = 1$, $\tau = 2$.

It is noted that when $s = 0$, the interior equilibrium point of the model system (6) is $P(\hat{x}, \hat{y}, \hat{z}, \hat{E}) = P(0.133, 0.026, 0.133, 3.993)$. Again, it is observed that when $s = -0.1$ the eigen values of the characteristic polynomial of the model system (6) are $-77.7359, -0.5, -0.00123587$ and the eigen values become $81.7046, -0.5, -0.00140128$ when $s = 0.1$.

Therefore it is clear from the above result that when s increases through zero two eigen values of the characteristic polynomial of the model system (6) remain same but one eigenvalue of the model system (6) moves from C^- to C^+ along the real axis by diverging through ∞. Hence, the stability of the model system (6) at the interior equilibrium point $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ changes from stable to unstable.
To stabilize the model system (6) in case of positive economic interest, let us consider a state feedback controller of the form $w(t) = u(E(t) - 3.993)$, consequently we have got the differential algebraic model system (8) as follows,

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{100}\right) - \frac{0.95xy}{30 + x} - 0.5Ex,$$

$$\frac{dy}{dt} = \frac{0.75yz}{30 + z} - 0.002y - 0.05y^2,$$

$$\frac{dz}{dt} = \frac{1}{2}(x - z),$$

$$(15 \times 0.5x - 1)E - s + u(E(t) - 3.993) = 0.$$

It is possible to evaluate the numerical value of net feedback gain from theorem 2. For the above model system we have got $u > \max(3.96146, 501.873, 420.653)$. Considering $u = 510$, we find the interior equilibrium point of the model system (8) as $(0.133642, 0.0265247, 0.133642, 3.99298)$ when $s = 0$ and the interior equilibrium point of the model system (8) becomes $(0.11794, 0.0187393, 0.11794, 3.9941)$ when $s = 0.1$.

It is evident from Figure1&2 that the differential algebraic model system (8) is clearly stable when s increases through zero i.e., singularity induced bifurcation phenomenon is eliminated from the differential algebraic model system (6) at the interior equilibrium point when net economic profit increases through zero.
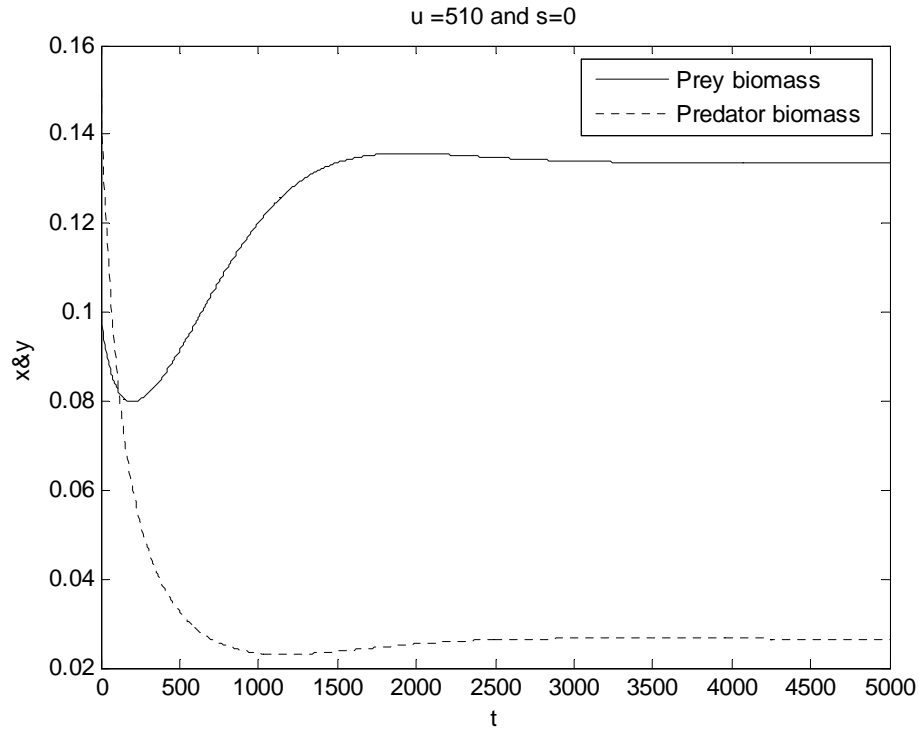
Figure 1. Variation of prey and predator biomass with the increasing time when $u = 510$ and $s = 0$.
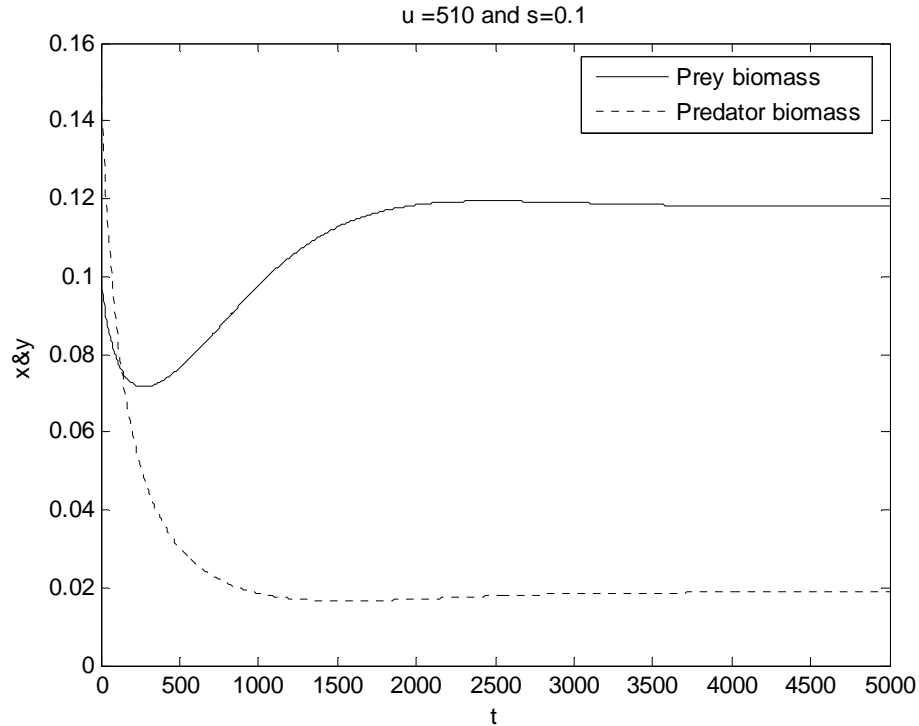
Figure 2. Variation of prey and predator biomass with the increasing time when u = 510 and s = 0.1.

5.2 Simulation when total economic profit is positive

In order to ensure the existence of Hopf bifurcation let us consider the parameters of the model system (6) as $\alpha = 0.6, r = 0.2, \beta = 0.08, K = 100, q = 0.05, a = 36, d = 0.0025, \gamma = 0.002, p = 12, c = 2, s = 2.$ Then the critical value of the bifurcation parameter $\tau^* = 7.4563.$ If we consider the value of $\tau = 7.2$ then it is observed from the figure1&2 that $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ is locally asymptotically stable and the populations x and y converge to their steady states in finite time. Now if we gradually increase the value of τ, keeping other parameters fixed, then by theorem3 we have got a critical value $\tau^* = 7.4563$ such that $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ loses its stability as τ passes through τ^*. Figure3&4 clearly show the result. It is also noted that if we consider the value of $\tau = 7.8$, then it is evident from figure5&6 that the positive equilibrium $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ is unstable and there is a periodic orbit near $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$.
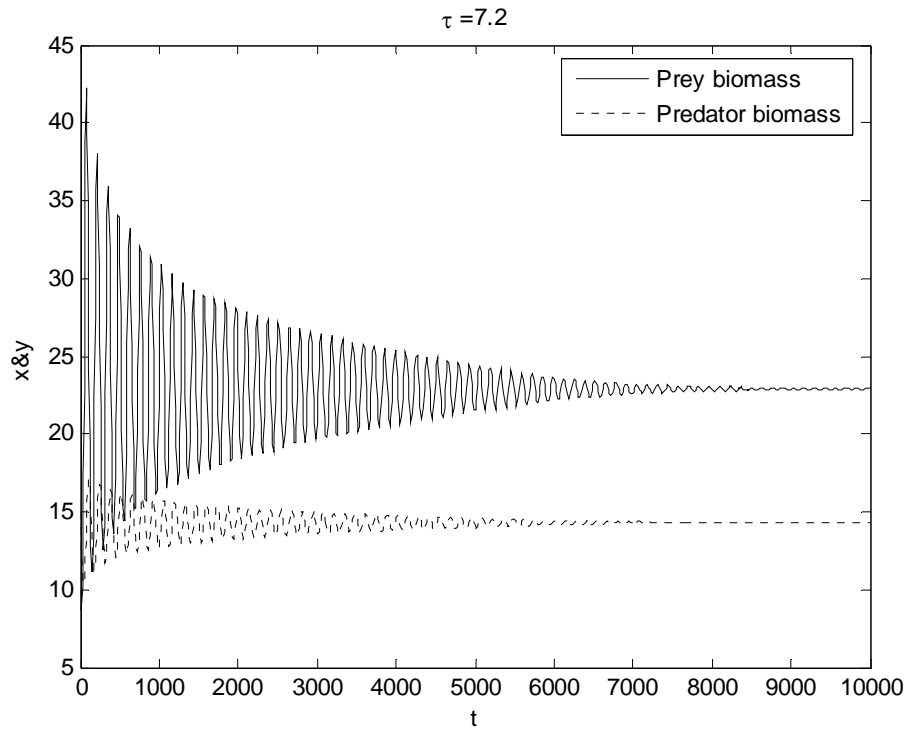
Figure 3. Variation of prey and predator biomass with the increasing time when $\tau = 7.2 < \tau^*$.
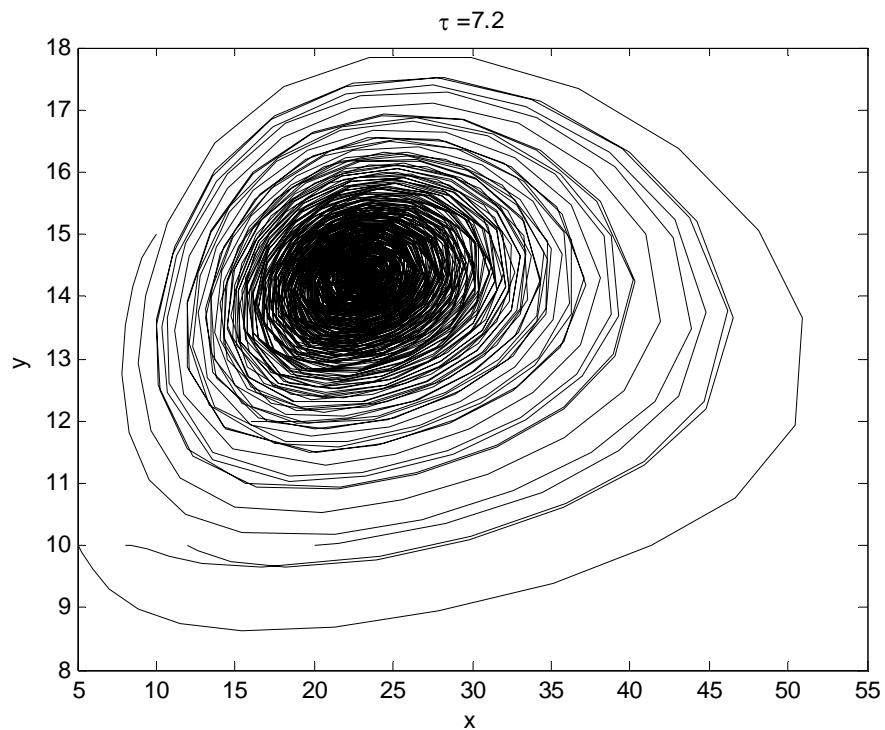
Figure 4. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 7.2 < \tau^*$.
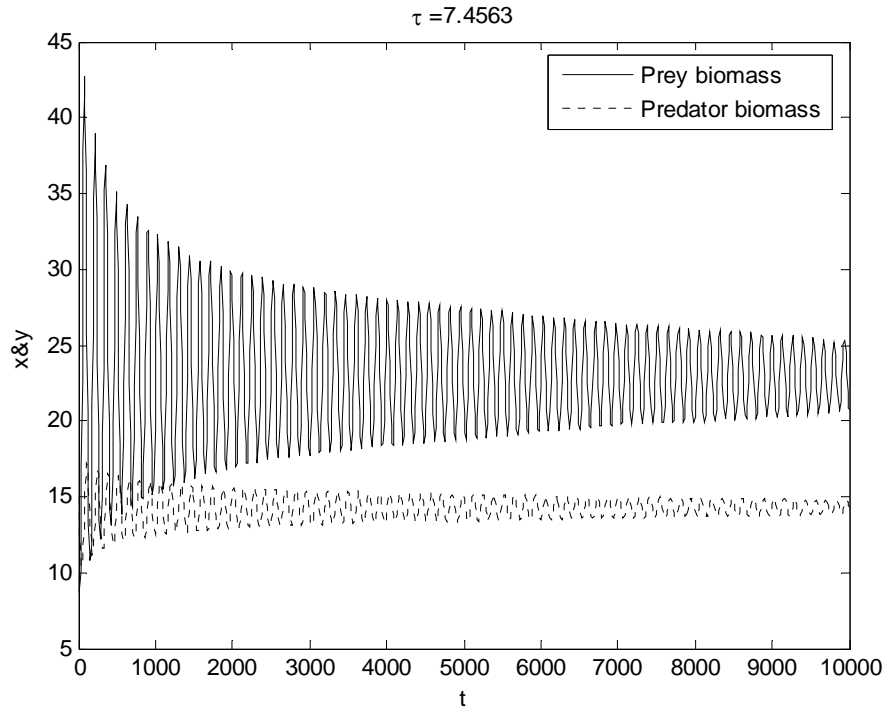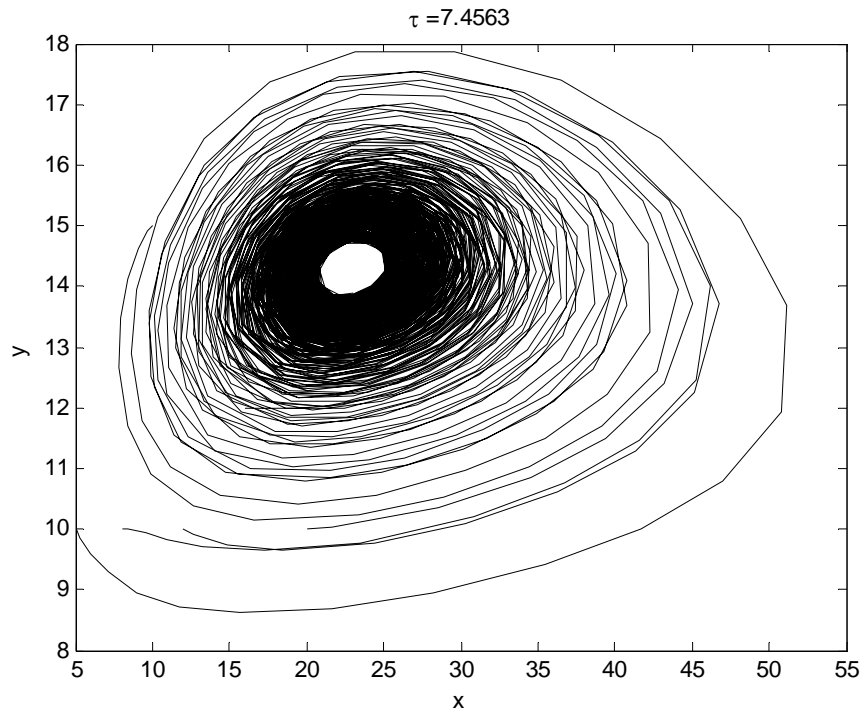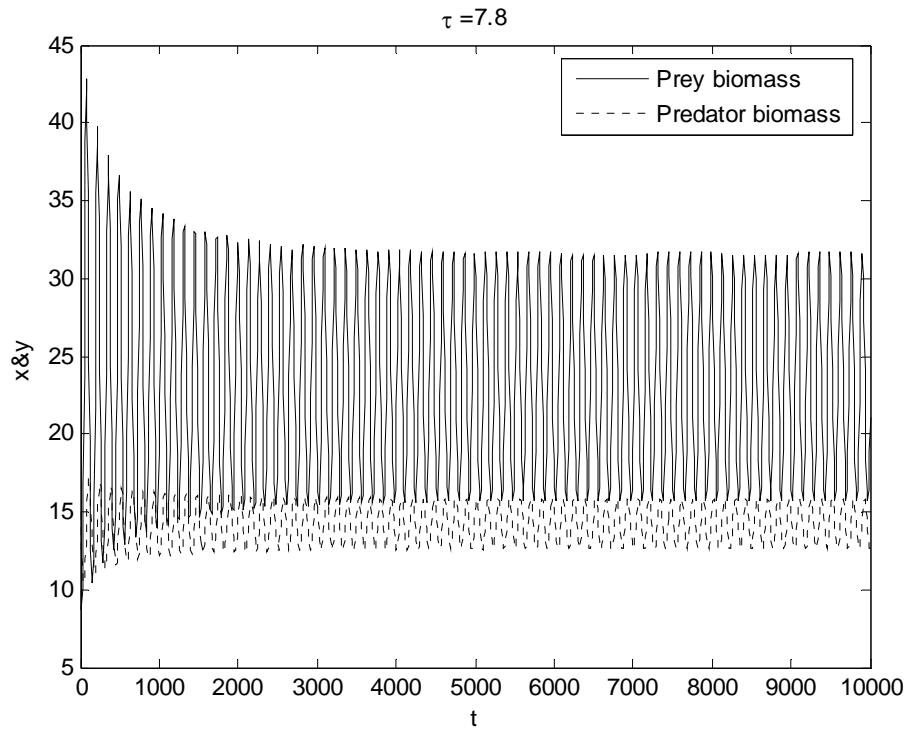
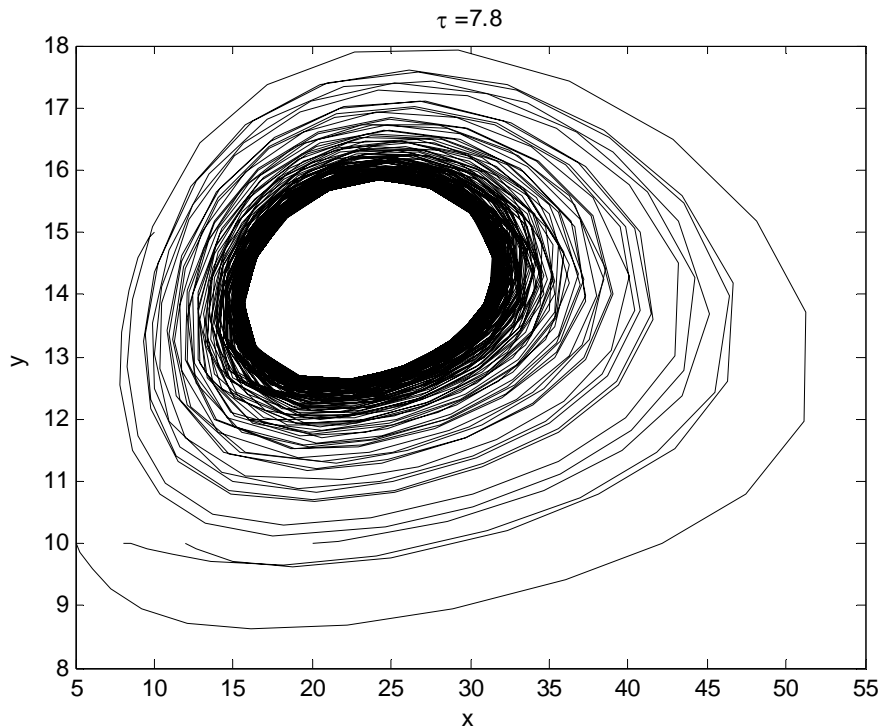Figure 5. Variation of prey and predator biomass with the increasing time when $\tau = 7.4563 = \tau^*$.

Figure 6. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 7.4563 = \tau^*$.

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Figure 7. Variation of prey and predator biomass with the increasing time when $\tau = 7.8 > \tau^*$.

Figure 8. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 7.8 > \tau^*$.

The aforesaid Hopf bifurcation can also be illustrated if we consider another set of numerical values to the parameters of the model system (6).

Let us consider the following set of parameters:

$\alpha = 0.8, r = 0.25, \beta = 0.08, K = 100, q = 0.5, a = 40, d = 0.003, \gamma = 0.002, p = 13.5, c = 3, s = 2.$

For this set of parameters the critical value of the bifurcation parameter $\tau^* = 10.678$. It is clearly observed that a simple Hopf bifurcation occurs at $\tau = \tau^*$ and for $\tau > \tau^*$ the interior equilibrium point, $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ approaches to a periodic solution.
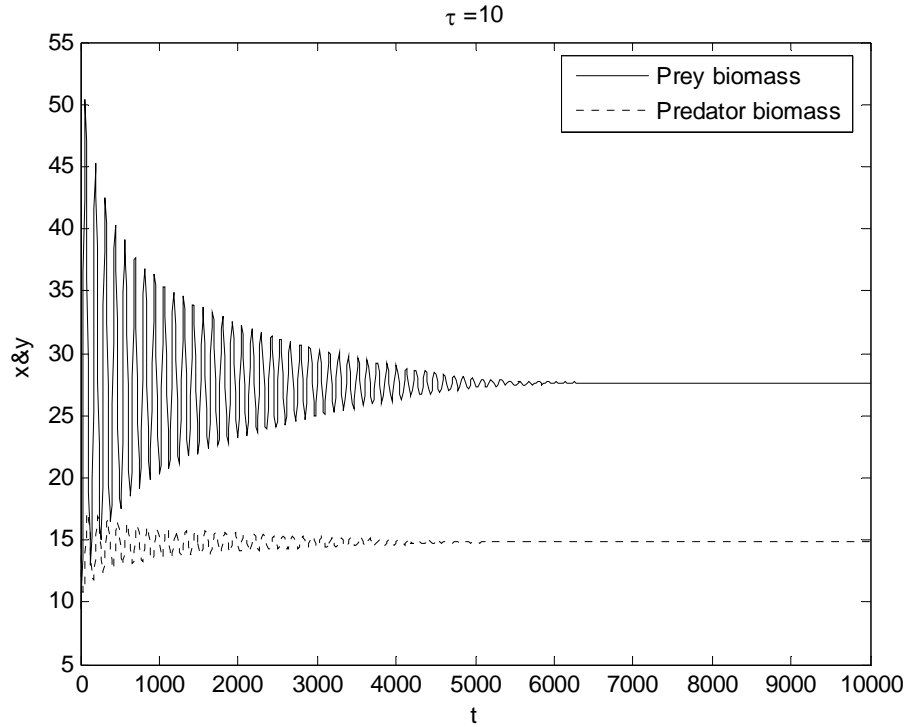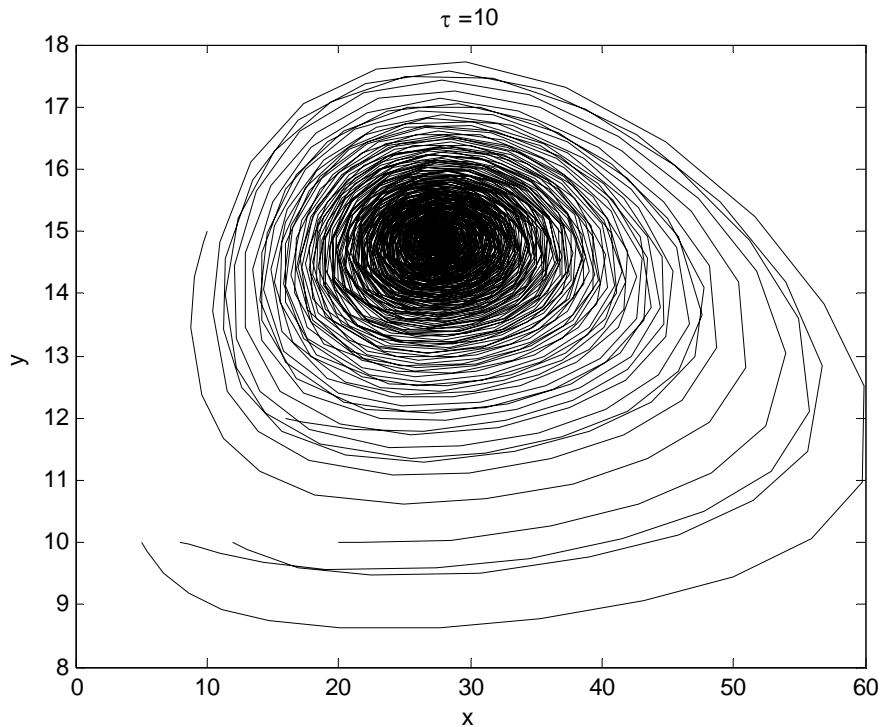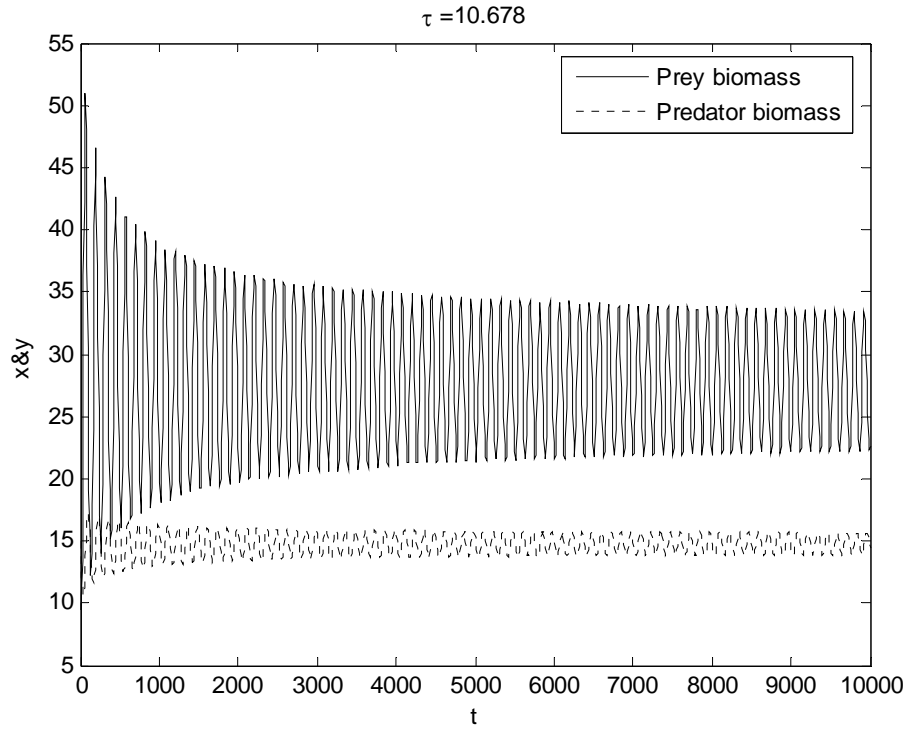

Figure 9. Variation of prey and predator biomass with the increasing time when $\tau = 10 < \tau^*$.

Figure 10. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 10 < \tau^*$.

$\tau = 10.678$

Figure 11. Variation of prey and predator biomass with the increasing time when $\tau = 10.678 = \tau^*$.
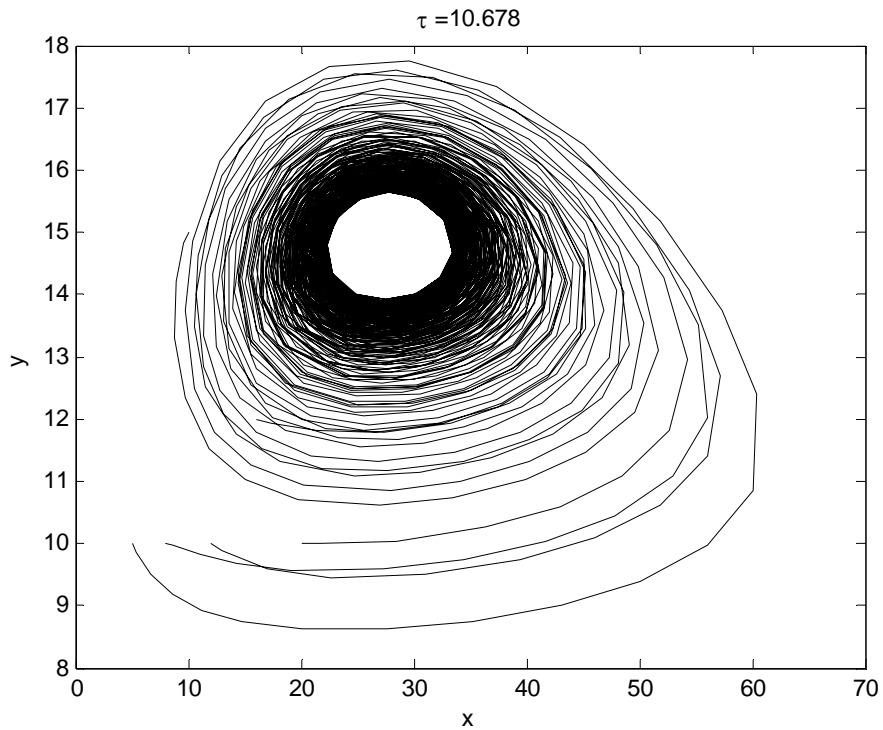
$\tau = 10.678$

Figure 12. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 10.678 = \tau^*$.
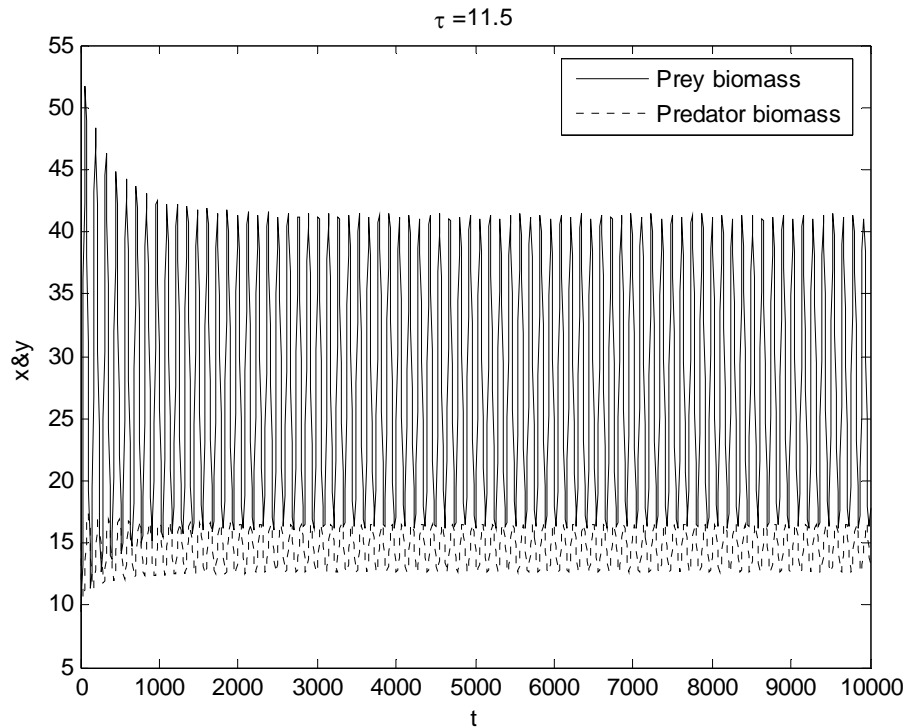
Figure 13. Variation of prey and predator biomass with the increasing time when $\tau = 11.5 > \tau^*$.
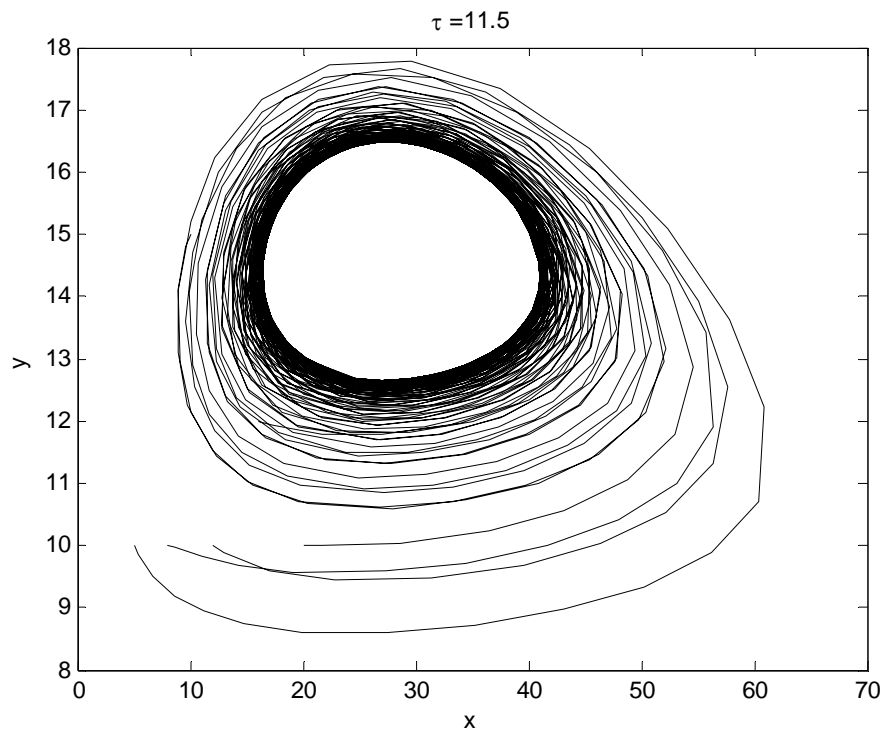

Figure 14. Phase space trajectories of prey and predator biomass beginning with different initial levels when $\tau = 11.5 > \tau^*$.

6. Concluding remarks

The paper analyzes the dynamical behavior of a prey predator model using differential-algebraic systems theory. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations thus we have studied the

effects of continuous time-delay on the dynamics of prey predator system. It is found that singularity induced bifurcation takes place when net economic revenue of the fishery is considered to be positive. In consequence to the aforesaid bifurcation, an impulsive phenomenon occurs and the system becomes unstable. The most important realistic feature of the paper is the state feedback controller which is designed to stabilize the model system when positive economic rent is taken into consideration. Numerical simulations are used to show that state feedback controller can be designed to resume the stability of a model system in case of positive economic profit. In the second part of the paper we have discussed the behavior of the model system with positive economic profit, here we have divided our discussion in two parts with and without time delay. In case of without time delay it is observed that though the model system is stable but it is possible to get a critical value of total economic profit so that the model system becomes unstable when total economic profit passes through the critical value and the model system enters into Hopf type small amplitude periodic solution. It is noted that continuous time delay also plays an important role to the dynamics of the model system. It is evident from the obtained results that the time delay can cause a stable equilibrium to become unstable and even a simple Hopf bifurcation occurs when the time delay passes through its critical value.

The entire study of the paper is mainly based on the deterministic framework. On the other hand it will be more realistic if it is possible to consider the model system in the stochastic environment due to some ecological fluctuations and other factors. Thus, a future research problem would be considered in stochastic environment. Again, to achieve the commercial purpose of the fishery it is also possible to determine optimal harvesting strategies using game theory.

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